COMPUTING K-THEORY AND EXT FOR GRAPH C^*-ALGEBRAS

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ABSTRACT. K-theory and Ext are computed for the C^*-algebra $C^*(E)$ of any countable directed graph E. The results generalize the K-theory computations of Raeburn and Szymański and the Ext computations of Tomforde for row-finite graphs. As a consequence, it is shown that if A is a countable $\{0, 1\}$ matrix and E_A is the graph obtained by viewing A as a vertex matrix, then $C^*(E_A)$ is not necessarily Morita equivalent to the Exel-Laca algebra $\mathcal{O}_A$.

1. INTRODUCTION

In [2] Cuntz and Krieger described a way to associate a C^*-algebra $\mathcal{O}_A$ to a finite square matrix A with entries in $\{0,1\}$. Since that time these Cuntz-Krieger algebras have been generalized in a remarkable number of ways. Perhaps the most direct of these is due to Exel and Laca, who define $\mathcal{O}_A$ for an infinite $\{0,1\}$-matrix A [4]. Another generalization involves associating a C^*-algebra to a countable directed graph. These graph algebras have drawn much interest because they comprise a wide class of C^*-algebras, and yet many of their C^*-algebraic properties can be easily deduced from the associated graphs.

In order to make sense of the relations for the generators of the graph algebra, it was often assumed in the original treatments that the graphs were row-finite; that is, each vertex is the source of finitely many edges [8, 7, 1]. However, in the past few years it has been shown how to define graph algebras for arbitrary graphs [5]. Consequently, much work has been done to extend results for the C^*-algebras of row-finite graphs to the C^*-algebras of arbitrary graphs [3, 5, 10, 11].

In [10], Raeburn and Szymański computed the K-theory of $C^*(E)$, where E is a row-finite directed graph. We briefly review that result here. Let J denote the set of sinks of E, let $I = E^0 \setminus J$, and let $A_E = \left(\begin{smallmatrix} B & C \\ 0 & 0 \end{smallmatrix}\right)$ denote the vertex matrix of E with respect to the decomposition $E^0 = I \cup J$. Then because E is row-finite, the matrix $\left(\begin{smallmatrix} B^t - I \\ C^t \end{smallmatrix}\right)$ determines a homomorphism from $\bigoplus_I \mathbb{Z}$ to $\bigoplus_I \mathbb{Z} \oplus \bigoplus_J \mathbb{Z}$. The kernel and cokernel of this homomorphism are isomorphic to $K_1(C^*(E))$ and $K_0(C^*(E))$, respectively. In [13], $\mathrm{Ext}(C^*(E))$

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is computed similarly for row-finite graphs E which satisfy Condition (L) and have no sinks. Specifically, $\mathrm{Ext}(C^*(E))$ is isomorphic to the cokernel of the homomorphism $A_E - I : \prod_{E^0} \mathbb{Z} \to \prod_{E^0} \mathbb{Z}$.

In this paper, we will show that the above results remain true for graphs which are not necessarily row-finite, provided we replace the word "sink" with the phrase "sink or vertex which emits infinitely many edges." We remark that Raeburn and Szymański have computed the K-theory for Exel-Laca algebras using direct limits [10, Theorem 4.1]. Also, the K-theory results have been obtained by Szymański in [12] for graphs with finitely many vertices, and the proof given there holds for arbitrary graphs as well. Our proof is different, and relies on desingularization [3], a tool for generalizing from the row-finite case to arbitrary graphs. If E is an arbitrary graph, we say a vertex v of E is a *singular vertex* if either v is a sink or v emits infinitely many edges. In [3], it is shown that there exists a graph F, called a *desingularization of E*, such that F has no singular vertices and $C^*(F)$ is Morita equivalent to $C^*(E)$. The key ingredient in our calculations of K-theory and Ext is a technical lemma, proven in Section 2, which shows that desingularizing a graph does not alter the kernel and cokernel of the maps determined by its vertex matrix. Thus we can apply the results of [10] and [13] to obtain the K-theory and Ext of $C^*(F)$ in terms of the vertex matrix of E. This, together with the fact that K-theory and Ext are stable, yields the K-theory and Ext of $C^*(E)$ stated in Theorem 3.1.

Finally, we use this result to shed some light on a question posed by Raeburn and Szymański in [10]. They showed that if A is any countable square $\{0, 1\}$-matrix and if E_A is the graph obtained by viewing A as a vertex matrix (that is, let E_A^0 be the index set of A and draw $A(i, j)$ edges from i to j), then the graph algebra $C^*(E_A)$ is a C^*-subalgebra of the Exel-Laca algebra $\mathcal{O}_A$. We will show that it is possible for $C^*(E_A)$ and $\mathcal{O}_A$ to have different K-theory. So in particular $C^*(E_A)$ is not always a full corner in $\mathcal{O}_A$.

2. The Technical Lemma

Given a graph E, it was shown in [3] how to construct a graph F, called a *desingularization of E*, such that F has no singular vertices and $C^*(E)$ is Morita equivalent to $C^*(F)$. We review that procedure here.

Definition 2.1. Suppose E is a graph with a singular vertex v_0. We *add a tail* to v_0 by performing the following procedure. List the vertices $w_0, w_1, \ldots$ of $r(s^{-1}(v_0))$. Note that the list of w's could be empty (if v_0 is a sink), finite, or countably infinite.

We begin by adding an infinite tail to v_0 as in [1, (1.2)]:

$$v_0 \xrightarrow{\;e_1\;} v_1 \xrightarrow{\;e_2\;} v_2 \xrightarrow{\;e_3\;} v_3 \cdots$$

Now, for every j with $w_j \in r(s^{-1}(v_0))$, let C_j be the number of edges from v_0 to w_j. For every i with $j \le i < j + C_j$, draw an edge labelled f_j^{i-j+1} from v_i to w_j. To be precise, if E is a graph with a singular vertex v_0, we define $F^0 := E^0 \cup \{v_1, v_2, \dots\}$ and

$$F^1 := \{e \in E^1 \mid s(e) \ne v_0\} \cup \{e_i\}_1^\infty \cup \bigcup_{\{j \mid w_j \in r(s^{-1}(v_0))\}} \{f_j^i\}_{i=1}^{C_j}.$$

We extend r and s to F as indicated above. In particular, $s(e_i) = v_{i-1}$, $r(e_i) = v_i$, $s(f_j^i) = v_{i+j-1}$, and $r(f_j^i) = w_j$.

Definition 2.2. If E is a directed graph, a *desingularization of E* is a graph F obtained by adding a tail at every singular vertex of E.

Note that different orderings of the vertices of $r(s^{-1}(v_0))$ may give rise to non-isomorphic graphs via the process of adding a tail. Thus a graph may have many desingularizations.

If E is a graph, then any desingularization F of E is a row-finite graph, so the rows of the matrix A_F are eventually zero. Thus $A_F : \prod_{F^0} \mathbb{Z} \to \prod_{F^0} \mathbb{Z}$ and $A_F^t : \bigoplus_{F^0} \mathbb{Z} \to \bigoplus_{F^0} \mathbb{Z}$.

Lemma 2.3. *Let E be a graph. Also let J be the set of singular vertices of E and let $I := E^0 \backslash J$. Then with respect to the decomposition $E^0 = I \cup J$ the vertex matrix of E will have the form*

$$A_E = \begin{pmatrix} B & C \\ * & * \end{pmatrix}$$

where B and C have entries in $\mathbb{Z}$ and the $*$'s have entries in $\mathbb{Z} \cup \{\infty\}$. If F is a desingularization of E, then $\operatorname{coker}(A_F - I) \cong \operatorname{coker}(B - I\ C)$ where $(B - I\ C) : \prod_I \mathbb{Z} \oplus \prod_J \mathbb{Z} \to \prod_I \mathbb{Z}$. Furthermore, $\ker(A_F^t - I) \cong \ker \begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$ and $\operatorname{coker}(A_F^t - I) \cong \operatorname{coker} \begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$, where $\begin{pmatrix} B^t - I \\ C^t \end{pmatrix} : \bigoplus_I \mathbb{Z} \to \bigoplus_I \mathbb{Z} \oplus \bigoplus_J \mathbb{Z}$.

Proof. List the elements of J as $J := \{v_1^0, v_2^0, v_3^0, \dots\}$. (Note that J may be either finite or countably infinite.) For each $1 \le i \le |J|$ let D_i be the $J \times \mathbb{N}$ matrix

$$D_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & \vdots & & & \ddots \end{pmatrix}$$

with a 1 in the $(i, 1)$ position and 0's elsewhere. Also let Z be the $\mathbb{N} \times \mathbb{N}$ matrix

$$Z = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ & \vdots & & & \ddots \end{pmatrix}$$

with -1's along the diagonal and 1's above the diagonal. Now for each $1 \le i \le |J|$ let $\{v_i^1, v_i^2, \dots\}$ be the vertices of the tail which is added to v_i^0 to form F. Then, by the way that desingularization is defined, we see that with respect to the decomposition $I \cup J \cup \{v_1^1, v_1^2, v_1^3, \dots\} \cup \{v_2^1, v_2^2, v_2^3, \dots\} \cup \dots$ the matrix $A_F - I$ will have the form

$$
A_F - I = \begin{pmatrix} B-I & C & 0 & 0 & \\ X_1 & Y_1-I & D_1 & D_2 & \cdots \\ X_2 & Y_2 & Z & 0 & \\ X_3 & Y_3 & 0 & Z & \\ & \vdots & & & \ddots \end{pmatrix}
$$

where the X_i's and Y_i's are row-finite. If we let $P := \prod_{\mathbb{N}} \mathbb{Z}$, then $A_F - I :$ $\prod_I \mathbb{Z} \oplus \prod_J \mathbb{Z} \oplus \prod_J P \to \prod_I \mathbb{Z} \oplus \prod_J \mathbb{Z} \oplus \prod_J P$. Also $(B-I\ C) : \prod_I \mathbb{Z} \oplus \prod_J \mathbb{Z} \to \prod_I \mathbb{Z}$. Let us define a map $\phi : \prod_I \mathbb{Z} \oplus \prod_J \mathbb{Z} \oplus \prod_J P \to \prod_I \mathbb{Z}$ by

$$
\phi\left(\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \end{pmatrix} \end{pmatrix} \right) = \mathbf{x}.
$$

We shall show that ϕ induces a map from $\mathrm{coker}(A_F - I)$ to $\mathrm{coker}(B - I\ C)$. Let

$$
\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \end{pmatrix} \end{pmatrix} = (A_F - I) \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \end{pmatrix} \end{pmatrix}.
$$

Then

$$
\phi\left(\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \end{pmatrix} \end{pmatrix} \right) = \mathbf{x} = (B - I\ C) \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \in \mathrm{im}(B - I\ C).
$$

Thus ϕ induces a map $\overline{\phi} : \mathrm{coker}(A_F - I) \to \mathrm{coker}(B - I\ C)$.

We shall show that $\overline{\phi}$ is an isomorphism. To see that $\overline{\phi}$ is injective suppose that

$$
\phi\left(\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \end{pmatrix} \end{pmatrix} \right) = \mathbf{x} \in \mathrm{im}(B - I\ C).
$$

Then there exists $\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \in \prod_I \mathbb{Z} \oplus \prod_J \mathbb{Z}$ such that $\mathbf{x} = (B - I)\mathbf{a} + C\mathbf{b}$. For each $1 \le i \le |J|$ let

$$
c_i^1 := \mathbf{y}_i - (X_1\mathbf{a} + (Y_1 - I)\mathbf{b})_i,
$$

where $(X_1\mathbf{a} + (Y_1 - I)\mathbf{b})_i$ denotes the ith entry of the vector $X_1\mathbf{a} + (Y_1 - I)\mathbf{b}$. Then, for each $k \in \{1, 2, \dots\}$ define c_i^k recursively by

$$c_i^{k+1} := c_i^k + (\mathbf{z}_i)_k - (X_{i+1}\mathbf{a} + Y_{i+1}\mathbf{b})_k,$$

where $(\mathbf{z}_i)_k$ denotes the kth entry of the vector $\mathbf{z}_i$ and $(X_{i+1}\mathbf{a} + Y_{i+1}\mathbf{b})_k$ denotes the kth entry of the vector $(X_{i+1}\mathbf{a} + Y_{i+1}\mathbf{b})$. Now for each $1 \le i \le |J|$ define $\mathbf{c}_i \in \prod_{\mathbb{N}} \mathbb{Z}$ by $\mathbf{c}_i := \begin{pmatrix} c_i^1 \\ c_i^2 \\ \vdots \end{pmatrix}$. Then

$$(A_F - I)\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \end{pmatrix} \end{pmatrix} = \begin{pmatrix} (B-I)\mathbf{a} + C\mathbf{b} \\ X_1\mathbf{a} + (Y_1 - I)\mathbf{b} + D_1\mathbf{c_1} + D_2\mathbf{c_2} + \dots \\ \begin{pmatrix} X_2\mathbf{a} + Y_2\mathbf{b} + Z\mathbf{c_1} \\ X_3\mathbf{a} + Y_3\mathbf{b} + Z\mathbf{c_2} \\ X_4\mathbf{a} + Y_4\mathbf{b} + Z\mathbf{c_3} \\ \vdots \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \end{pmatrix} \end{pmatrix}$$

and thus $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \end{pmatrix} \end{pmatrix} \in \text{im}(A_F - I)$ and $\overline{\phi}$ is injective. Furthermore, since ϕ is

surjective it follows that $\overline{\phi}$ is surjective. Thus $\text{coker}(A_F - I) \cong \text{coker}(B - I \; C)$.

Next we shall examine $A_F^t - I$. Note that with respect to the decomposition mentioned earlier $A_F^t - I$ will have the form

$$A_F^t - I = \begin{pmatrix} B^t - I & X_1^t & X_2^t & X_3^t & \\ C^t & Y_1^t - I & Y_2^t & Y_3^t & \cdots \\ 0 & D_1^t & Z^t & 0 & \\ 0 & D_2^t & 0 & Z^t & \\ & \vdots & & & \ddots \end{pmatrix}$$

where the X_i^t's and Y_i^t's are column-finite matrices. If we let $Q := \bigoplus_{\mathbb{N}} \mathbb{Z}$, then $(A_F^t - I) : \bigoplus_I \mathbb{Z} \oplus \bigoplus_J \mathbb{Z} \oplus \bigoplus Q \to \bigoplus_I \mathbb{Z} \oplus \bigoplus_J \mathbb{Z} \oplus \bigoplus Q$. Also $\begin{pmatrix} B^t - I \\ C^t \end{pmatrix} : \bigoplus_I \mathbb{Z} \to \bigoplus_I \mathbb{Z} \oplus \bigoplus_J \mathbb{Z}$. Let us define a map $\psi : \bigoplus_I \mathbb{Z} \to \bigoplus_I \mathbb{Z} \oplus \bigoplus_J \mathbb{Z} \oplus \bigoplus Q$ by

$$\Psi(\mathbf{x}) = \begin{pmatrix} \mathbf{x} \\ 0 \\ 0 \end{pmatrix}.$$

Note that if $\mathbf{x} \in \ker \begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$, then

$$(A_F^t - I)\begin{pmatrix} \mathbf{x} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (B^t - I)\mathbf{x} \\ C^t\mathbf{x} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so ψ restricts to a map $\psi : \ker \begin{pmatrix} B^t - I \\ C^t \end{pmatrix} \to \ker(A_F^t - I)$. We shall show that this map is surjective. Suppose that

$$\left(\begin{array}{c} \mathbf{x} \\ \mathbf{y} \\ \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \end{pmatrix} \end{array} \right) \in \ker(A_F^t - I).$$

Then for each $1 \le i \le |J|$ we must have that $D_i^t \mathbf{y} + Z^t \mathbf{z}_i = 0$. If $\mathbf{z}_i = \begin{pmatrix} z_i^1 \\ z_i^2 \\ \vdots \end{pmatrix}$,

then for all $k \in \mathbb{N}$ we must have

$$y_i - z_i^1 = 0 \quad \text{and} \quad z_i^k - z_i^{k+1} = 0.$$

Since $\mathbf{z}_i \in \bigoplus_J \mathbb{Z}$ we know that z_i^k is eventually zero. Thus the above equations imply that $y_i = z_i^1 = z_i^2 = \ldots = 0$. Since this holds for all i we have that

$$\left(\begin{array}{c} \mathbf{x} \\ \mathbf{y} \\ \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \end{pmatrix} \end{array} \right) = \begin{pmatrix} \mathbf{x} \\ 0 \\ 0 \end{pmatrix} = \psi(\mathbf{x})$$

and ψ is surjective. Furthermore, since ψ is clearly injective, $\psi : \ker \begin{pmatrix} B^t - I \\ C^t \end{pmatrix} \to$

$\ker(A_F^t - I)$ is an isomorphism and $\ker \begin{pmatrix} B^t - I \\ C^t \end{pmatrix} \cong \ker(A_F^t - I)$.

Next we shall define a map $\rho : \bigoplus_I \mathbb{Z} \oplus \bigoplus_J \mathbb{Z} \to \bigoplus_I \mathbb{Z} \oplus \bigoplus_J \mathbb{Z} \oplus \bigoplus_J Q$ by

$$\rho \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ 0 \end{pmatrix}.$$

We shall show that ρ induces a map from $\mathrm{coker} \begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$ to $\mathrm{coker}(A_F - I)$.

Suppose that $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathrm{im} \begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$. Then there exists an element $\mathbf{a} \in \bigoplus_I \mathbb{Z}$

such that $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} (B^t - I)\mathbf{a} \\ C^t \mathbf{a} \end{pmatrix}$. Hence

$$(A_F^t - I) \begin{pmatrix} \mathbf{a} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (B^t - I)\mathbf{a} \\ C^t \mathbf{a} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ 0 \end{pmatrix}.$$

Thus ρ maps $\mathrm{im}\begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$ into $\mathrm{im}(A_F^t - I)$ and hence induces a map $\overline{\rho}$: $\mathrm{coker}\begin{pmatrix} B^t - I \\ C^t \end{pmatrix} \to \mathrm{coker}(A_F^t - I)$. We shall show that this map is injective.

Suppose that $\overline{\rho}\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$ equals zero in $\mathrm{coker}(A_F^t - I)$. Then

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ 0 \end{pmatrix} = (A_F^t - I)\left(\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \end{pmatrix} \end{pmatrix} \right) \quad \text{for some} \quad \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \end{pmatrix} \end{pmatrix} \in \bigoplus_I \mathbb{Z} \oplus \bigoplus_J \mathbb{Z} \oplus \bigoplus_J Q.$$

But then as before we must have that $\mathbf{b} = \mathbf{c}_1 = \mathbf{c}_2 = \ldots = 0$ and the above equation implies that $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} (B^t - I)\mathbf{a} \\ C^t \mathbf{a} \end{pmatrix} \in \mathrm{im}\begin{pmatrix} B^t - I \\ C^t \end{pmatrix}$ so $\overline{\rho}$ is injective.

We shall now show that $\overline{\rho}$ is surjective. Let $\left(\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \end{pmatrix} \end{pmatrix} \right) \in \bigoplus_I \mathbb{Z} \oplus \bigoplus_J \mathbb{Z} \oplus \bigoplus_J Q.$

It suffices to show that there exists $\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ 0 \end{pmatrix} \in \bigoplus_I \mathbb{Z} \oplus \bigoplus_J \mathbb{Z} \oplus \bigoplus_J Q$ such that

$$\left(\begin{pmatrix} \mathbf{x} - \mathbf{u} \\ \mathbf{y} - \mathbf{v} \\ \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \end{pmatrix} \end{pmatrix} \right) \in \mathrm{im}(A_F^t - I).$$

For each $1 \leq i \leq |J|$ write $\mathbf{z}_i = \begin{pmatrix} z_i^1 \\ z_i^2 \\ \vdots \end{pmatrix}$ and define

$$b_i := \sum_{j=1}^{\infty} z_i^j \qquad \text{and} \qquad c_i^k := \sum_{j=k+1}^{\infty} z_i^j \ \text{ for } k \in \mathbb{N}.$$

Note that since $\mathbf{z}_i$ is in the direct sum, all of the above sums are finite, and since $\begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \end{pmatrix} \in \bigoplus_J Q$ we have that eventually $\mathbf{z}_i = 0$ and hence

$$\mathbf{b} := \begin{pmatrix} b_1 \\ b_2 \\ \vdots \end{pmatrix} \in \bigoplus_J \mathbb{Z} \ \text{ and } \ \mathbf{c} := \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \end{pmatrix} \in \bigoplus_J Q, \text{ where } \mathbf{c}_i := \begin{pmatrix} c_i^1 \\ c_i^2 \\ \vdots \end{pmatrix}.$$

If we then take

$$\mathbf{u} := \mathbf{x} - (B^t - I)\mathbf{a} - X_1^t \mathbf{b} - X_2^t \mathbf{c}_1 - X_3^t \mathbf{c}_2 - \ldots$$

and
$$\mathbf{v} := \mathbf{y} - C^t\mathbf{a} - (Y_1^t - I)\mathbf{b} - Y_2^t\mathbf{c}_1 - Y_3^t\mathbf{c}_2\ldots,$$

which are finite sums since $\begin{pmatrix}\mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots\end{pmatrix}$ is in the direct sum, we have that

$$(A_F^t - I)\begin{pmatrix}\mathbf{a} \\ \mathbf{b} \\ \begin{pmatrix}\mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\mathbf{x} - \mathbf{u} \\ \mathbf{y} - \mathbf{v} \\ \begin{pmatrix}\mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots\end{pmatrix}\end{pmatrix}.$$

Thus $\overline{\rho}$ is surjective. Hence $\overline{\rho}$ is an isomorphism and $\operatorname{coker}\begin{pmatrix}B^t - I \\ C^t\end{pmatrix} \cong$
$\operatorname{coker}(A_F^t - I)$. $\qquad\square$

3. MAIN RESULTS

Theorem 3.1. *Let E be a graph. Also let J be the set of singular vertices of E and let $I := E^0 \backslash J$. Then with respect to the decomposition $E^0 = I \cup J$ the vertex matrix of E will have the form*

$$A_E = \begin{pmatrix}B & C \\ * & *\end{pmatrix}$$

where B and C have entries in $\mathbb{Z}$ and the $*$'s have entries in $\mathbb{Z} \cup \{\infty\}$. Then $K_0(C^*(E)) \cong \operatorname{coker}\begin{pmatrix}B^t - I \\ C^t\end{pmatrix}$ and $K_1(C^*(E)) \cong \ker\begin{pmatrix}B^t - I \\ C^t\end{pmatrix}$ where $\begin{pmatrix}B^t - I \\ C^t\end{pmatrix} : \bigoplus_I \mathbb{Z} \to \bigoplus_I \mathbb{Z} \oplus \bigoplus_J \mathbb{Z}$.

If, in addition, E satisfies Condition (L), then $\operatorname{Ext}(C^*(E)) \cong \operatorname{coker}(B - I \ C)$ where $(B - I \ C) : \prod_I \mathbb{Z} \oplus \prod_J \mathbb{Z} \to \prod_I \mathbb{Z}$.

Proof. Let F be a desingularization of E. Since F is row-finite and has no sinks it follows from [10, Theorem 3.2] that $K_0(C^*(E)) \cong \operatorname{coker}(A_F^t - I)$ and $K_1(C^*(E)) \cong \ker(A_F^t - I)$. By [3, Theorem 2.11] $C^*(E)$ is Morita equivalent to $C^*(F)$. Because K-theory is stable, we have that $K_0(C^*(E)) \cong \operatorname{coker}(A_F^t - I)$ and $K_1(C^*(E)) \cong \ker(A_F^t - I)$. The result then follows from Lemma 2.3.

Furthermore, if E satisfies Condition (L), then it follows from [3, Lemma 2.7] that F also satisfies Condition (L). Hence by [13, Theorem 6.16] we have that $\operatorname{Ext}(C^*(F)) \cong \operatorname{coker}(B - I \ C)$. Since Ext is stable, the result again follows from Lemma 2.3. $\qquad\square$

Corollary 3.2. *If every vertex of E is either a sink or emits infinitely many edges, then $K_0(C^*(E)) \cong \bigoplus_{E^0} \mathbb{Z}$ and $K_1(C^*(E)) \cong \operatorname{Ext}(C^*(E)) \cong \{0\}$.*

Proof. $I = \emptyset$, so we have $\bigoplus_I \mathbb{Z} = \prod_I \mathbb{Z} = \{0\}$, and the result then follows from Theorem 3.1. $\qquad\square$

In [10], Raeburn and Szymański prove that every graph algebra is an Exel-Laca algebra, but not conversely. In particular, they produce a matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 & & \\ 0 & 1 & 1 & 1 & 1 & 1 & & \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & \\ 0 & 1 & 0 & 1 & 0 & 0 & & \\ 0 & 0 & 1 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & 0 & 1 & & \\ & & \vdots & & & & \ddots & \end{pmatrix}$$

such that the Exel-Laca algebra $\mathcal{O}_A$ is not a graph algebra. They do prove, however, that $C^*(E_A)$ is a C^*-subalgebra in $\mathcal{O}_A$, where E_A is the graph whose vertex matrix is A [10, Proposition 5.1], and this prompts them to ask if anything more can be said about the relationship between the two.

It appears not. For if A and E_A are as above, the reader can check using Theorem 3.1 that $K_0(C^*(E_A)) \cong K_1(C^*(E_A)) \cong \{0\}$. In [10, Remark 4.3], the K-theory of $\mathcal{O}_A$ is computed as $K_0(\mathcal{O}_A) \cong \{0\}$ and $K_1(\mathcal{O}_A) \cong \mathbb{Z}$. Hence $C^*(E_A)$ is not a full corner of $\mathcal{O}_A$, and in fact $C^*(E_A)$ and $\mathcal{O}_A$ are not even Morita equivalent.

We also point out that, for the matrix A above, knowing the K-theory of $C^*(E_A)$ allows one to actually determine $C^*(E_A)$ up to isomorphism. $C^*(E_A)$ is a purely infinite, simple, separable, nuclear C^*-algebra without unit and hence the Kirchberg-Phillips Classification Theorem tells us that it it is determined up to Morita equivalence by its K-theory [9, Theorem 4.2.4]. Since $\mathcal{O}_2$ has the same K-theory we may conclude that $C^*(E_A)$ is Morita equivalent to $\mathcal{O}_2$. Finally, since E_A is transitive with infinitely many vertices it follows from [6, Theorem 2.13] that $C^*(E_A)$ is stable. Hence $C^*(E_A) \cong \mathcal{O}_2 \otimes \mathcal{K}$.

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